CAYLEY GRAPHS OF ORDER $27p$ ARE HAMILTONIAN

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Abstract. Suppose G is a finite group, such that $|G| = 27p$, where p is prime. We show that if S is any generating set of G, then there is a hamiltonian cycle in the corresponding Cayley graph $\mathrm{Cay}(G; S)$.

Theorem. *If $|G| = 27p$, where p is prime, then every connected Cayley graph on G has a hamiltonian cycle.*

Combining this with results in [1, 2, 6] establishes that:

$$\text{(0.1)} \qquad \begin{array}{c} \textit{Every Cayley graph on } G \textit{ has a hamiltonian cycle} \\ \text{if } |G| = kp, \text{ where } p \text{ is prime, } 1 \leq k < 32, \text{ and } k \neq 24. \end{array}$$

The remainder of the paper provides a proof of the theorem. Here is an outline:

1. Preliminaries: known results on hamiltonian cycles in Cayley graphs

For convenience, we record some known results that provide hamiltonian cycles in various Cayley graphs, after fixing some notation.

Notation [3, §1.1 and §5.1]**.** For any group G, we use:

(1) G' to denote the *commutator subgroup* $[G, G]$ of G,
(2) $Z(G)$ to denote the *center* of G, and
(3) $\Phi(G)$ to denote the *Frattini subgroup* of G.

For $a, b \in G$, we use a^b to denote the *conjugate* $b^{-1}ab$.

Notation. If $(s_1, s_2, \ldots, s_n)$ is any sequence, we use $(s_1, s_2, \ldots, s_n)\#$ to denote the sequence $(s_1, s_2, \ldots, s_{n-1})$ that is obtained by deleting the last term.

(1.1) **Theorem** (Marušič, Durnberger, Keating-Witte [5])**.** *If G' is a cyclic group of prime-power order, then every connected Cayley graph on G has a hamiltonian cycle.*

(1.2) **Lemma** [6, Lem. 2.27]**.** *Let S generate the finite group G, and let $s \in S$. If*

- $\langle s \rangle \triangleleft G$,

- $\operatorname{Cay}\bigl(G/\langle s\rangle; S\bigr)$ *has a hamiltonian cycle, and*
- *either*
  (1) $s \in Z(G)$*, or*
  (2) $|s|$ *is prime,*

then $\operatorname{Cay}(G;S)$ *has a hamiltonian cycle.*

(1.3) **Lemma** [1, Lem. 2.7]**.** *Let S generate the finite group G, and let $s \in S$. If*

- $\langle s\rangle \triangleleft G$*,*
- $|s|$ *is a divisor of pq, where p and q are distinct primes,*
- $s^p \in Z(G)$*,*
- $|G/\langle s\rangle|$ *is divisible by q, and*
- $\operatorname{Cay}\bigl(G/\langle s\rangle; S\bigr)$ *has a hamiltonian cycle,*

then there is a hamiltonian cycle in $\operatorname{Cay}(G;S)$*.*

The following results are well known (and easy to prove):

(1.4) **Lemma** ("Factor Group Lemma")**.** *Suppose*

- *S is a generating set of G,*
- *N is a cyclic, normal subgroup of G,*
- *$(s_1N, \ldots, s_nN)$ is a hamiltonian cycle in* $\operatorname{Cay}(G/N;S)$*, and*
- *the product $s_1 s_2 \cdots s_n$ generates N.*

Then $(s_1, \ldots, s_n)^{|N|}$ is a hamiltonian cycle in $\operatorname{Cay}(G;S)$*.*

(1.5) **Corollary.** *Suppose*

- *S is a generating set of G,*
- *N is a normal subgroup of G, such that $|N|$ is prime,*
- *$s \equiv t \pmod N$ for some $s, t \in S \cup S^{-1}$ with $s \neq t$, and*
- *there is a hamiltonian cycle in* $\operatorname{Cay}(G/N;S)$ *that uses at least one edge labelled s.*

Then there is a hamiltonian cycle in $\operatorname{Cay}(G;S)$*.*

(1.6) **Definition.** If H is any subgroup of G, then $H\backslash\operatorname{Cay}(G;S)$ denotes the multigraph in which:

- the vertices are the right cosets of H, and
- there is an edge joining Hg_1 and Hg_2 for each $s \in S \cup S^{-1}$, such that $g_1 s \in Hg_2$.

Thus, if there are two different elements s_1 and s_2 of $S \cup S^{-1}$, such that $g_1 s_1$ and $g_1 s_2$ are both in Hg_2, then the vertices Hg_1 and Hg_2 are joined by a double edge.

(1.7) **Lemma** [6, Cor. 2.9]**.** *Suppose*

- *S is a generating set of G,*
- *H is a subgroup of G, such that $|H|$ is prime,*
- *the quotient multigraph $H\backslash\operatorname{Cay}(G;S)$ has a hamiltonian cycle C, and*
- *C uses some double-edge of $H\backslash\operatorname{Cay}(G;S)$.*

Then there is a hamiltonian cycle in $\operatorname{Cay}(G;S)$*.*

(1.8) **Theorem** [7, Cor. 3.3]**.** *Suppose*

- *S is a generating set of G,*
- *N is a normal p-subgroup of G, and*
- *$st^{-1} \in N$, for all $s, t \in S$.*

Then $\mathrm{Cay}(G;S)$ *has a hamiltonian cycle.*

(1.9) *Remark.* In the proof of our main result, we may assume $p \geq 5$, for otherwise either:

- $|G| = 54$ is of the form $18q$, where q is prime, so [6, Prop. 9.1] applies, or
- $|G| = 3^4$ is a prime power, so the main theorem of [8] applies.

2. Assume the Sylow p-subgroup of G is normal

Notation. Let:

- G be a group of order $27p$, where p is prime, and $p \geq 5$ (see Remark 1.9),
- S be a minimal generating set for G,
- $P \cong \mathbb{Z}_p$ be a Sylow p-subgroup of G,
- w be a generator of P, and
- Q be a Sylow 3-subgroup of G

(2.1) **Assumption.** *In this section, we assume that P is a normal subgroup of G.*

Therefore G is a semidirect product:

$$G = Q \ltimes P.$$

We may assume G' is not cyclic of prime order (for otherwise Theorem 1.1 applies). This implies Q is nonabelian, and acts nontrivially on P, so

$$G' = Q' \times P \text{ is cyclic of order } 3p.$$

Notation. Since Q is a 3-group and acts nontrivially on $P \cong \mathbb{Z}_p$, we must have $p \equiv 1 \pmod 3$. Thus, we may choose $r \in \mathbb{Z}$, such that

$$r^3 \equiv 1 \pmod p, \quad \text{but } r \not\equiv 1 \pmod p.$$

Dividing $r^3 - 1$ by $r - 1$, we see that

$$r^2 + r + 1 \equiv 0 \pmod p.$$

2A. **A lemma that applies to both of the possible Sylow 3-subgroups.** There are only 2 nonabelian groups of order 27, and we will consider them as separate cases, but, first, we cover some common ground.

(2.2) **Note.** Since Q is a nonabelian group of order 27, and $G = Q \ltimes P \cong Q \ltimes \mathbb{Z}_p$, it is easy to see that

$$Q' = \Phi(Q) = Z(Q) = Z(G) = \Phi(G).$$

(2.3) **Lemma.** *Assume*

- $s \in (S \cup S^{-1}) \cap Q$*, such that s does not centralize P, and*
- $c \in C_Q(P) \smallsetminus \Phi(Q)$.

Then we may assume S is either $\{s, cw\}$ or $\{s, c^2w\}$ or $\{s, scw\}$ or $\{s, sc^2w\}$.

Proof. Since $G/P \cong Q$ is a 2-generated group of prime-power order, there must be an element a of S, such that $\{s, a\}$ generates G/P. We may write

$$a = s^i c^j z w^k, \quad \text{with } 0 \leq i \leq 2,\ 1 \leq j \leq 2,\ z \in Z(Q), \text{ and } 0 \leq k < p.$$

Note that:

- By replacing a with its inverse if necessary, we may assume $i \in \{0, 1\}$.

- By applying an automorphism of G that fixes s and maps c to cz^j, we may assume z is trivial (since $(cz^j)^j = c^j z^{j^2} = c^j z$).
- By replacing w with w^k if $k \neq 0$, we may assume $k \in \{0,1\}$.

Thus,

$$a = s^i c^j w^k \quad \text{with } i,k \in \{0,1\} \text{ and } j \in \{1,2\}.$$

Case 1. *Assume* $k=1$. Then $\langle s,a\rangle = G$, so $S = \{s,a\}$. This yields the four listed generating sets.

Case 2. *Assume* $k=0$. Then $\langle s,a\rangle = Q$, and there must be a third element b of S, with $b \notin Q$; after replacing w with an appropriate power, we may write $b = tw$ with $t \in Q$. We must have $t \in \langle s, \Phi(Q)\rangle$, for otherwise $\langle s,b\rangle = G$ (which contradicts the minimality of S). Therefore

$$t = s^{i'} z' \text{ with } 0 \le i' \le 2 \text{ and } z' \in \Phi(Q) = Z(G).$$

We may assume:

- $i' \neq 0$, for otherwise $b = z'w \in S \cap \big(Z(G) \times P\big)$, so Lemma 1.3 applies.
- $i' = 1$, by replacing b with its inverse if necessary.
- $z' \neq e$, for otherwise s and b provide a double edge in $\mathrm{Cay}(G/P;S)$, so Corollary 1.5 applies.

Then $s^{-1}b = z'w$ generates $Z(G) \times P$.

Consider the hamiltonian cycles

$$(a^{-1},s^2)^3, \quad \big((a^{-1},s^2)^3\#, b\big), \quad \text{and} \quad \big((a^{-1},s^2)^3\#\#, b^2\big)$$

in $\mathrm{Cay}(G/\langle z,w\rangle;S)$. Letting $z'' = (a^{-1}s^2)^3 \in \langle z\rangle$, we see that their endpoints in G are (respectively):

$$z'', \quad z''(s^{-1}b) = z''z'\,w, \quad \text{and} \quad z''(s^{-1}b)^s(s^{-1}b) = z''(z')^2\,w^s w.$$

The final two endpoints both have a nontrivial projection to P (since s, being a 3-element, cannot invert w), and at least one of these two endpoints also has a nontrivial projection to $Z(G)$. Such an endpoint generates $Z(G) \times P = \langle z,w\rangle$, so the Factor Group Lemma (1.4) provides a hamiltonian cycle in $\mathrm{Cay}(G;S)$. □

2B. Sylow 3-subgroup of exponent 3.

(2.4) **Lemma.** *Assume* Q *is of exponent* 3*, so*

$$Q = \langle\, x,y,z \mid x^3 = y^3 = z^3 = e,\ [x,y] = z,\ [x,z] = [y,z] = e \,\rangle.$$

Then we may assume:

(1) $w^x = w^r$*, but* y *and* z *centralize* P*, and*

(2) *either:*

  (a) $S = \{x, yw\}$*, or*

  (b) $S = \{x, xyw\}$.

Proof. (1) Since Q acts nontrivially on P, and $\mathrm{Aut}(P)$ is cyclic, but $Q/\Phi(Q)$ is not cyclic, there must be elements a and b of $Q \setminus \Phi(Q)$, such that a centralizes P, but b does not. (And z must centralize P, because it is in Q'.) By applying an automorphism of Q, we may assume $a = y$ and $b = x$. Furthermore, we may assume $w^x = w^r$ by replacing x with its inverse if necessary.

(2) S must contain an element that does not centralize P, so we may assume $x \in S$. By applying Lemma 2.3 with $s = x$ and $c = y$, we see that we may assume S is:

$$\{x, yw\} \text{ or } \{x, y^2w\} \text{ or } \{x, xyw\} \text{ or } \{x, xy^2w\}.$$

But there is an automorphism of G that fixes x and w, and sends y to y^2, so we need only consider 2 of these possibilities. □

(2.5) **Proposition.** *Assume, as usual, that $|G| = 27p$, where p is prime, and that G has a normal Sylow p-subgroup. If the Sylow 3-subgroup Q is of exponent 3, then $\mathrm{Cay}(G;S)$ has a hamiltonian cycle.*

Proof. We write $\overline{}$ for the natural homomorphism from G to $\overline{G} = G/P$. From Lemma 2.4(2), we see that we need only consider two possibilities for S.

Case 1. *Assume $S = \{x, yw\}$.* For $a = x$ and $b = yw$, we have the following hamiltonian cycle in $\mathrm{Cay}(G/P;S)$:

$$\begin{array}{cccccccccccccc}
 & \overline{e} & \xrightarrow{a} & \overline{x} & \xrightarrow{a} & \overline{x^2} & \xrightarrow{b} & \overline{x^2y} & \xrightarrow{a^{-1}} & \overline{xyz} & \xrightarrow{a^{-1}} & \overline{yz^2} \\
\xrightarrow{b} & \overline{y^2z^2} & \xrightarrow{b} & \overline{z^2} & \xrightarrow{a} & \overline{xz^2} & \xrightarrow{a} & \overline{x^2z^2} & \xrightarrow{b} & \overline{x^2yz^2} & \xrightarrow{a} & \overline{yz} \\
\xrightarrow{a} & \overline{xy} & \xrightarrow{b} & \overline{xy^2} & \xrightarrow{a} & \overline{x^2y^2z} & \xrightarrow{b} & \overline{x^2z} & \xrightarrow{b} & \overline{x^2yz} & \xrightarrow{a^{-1}} & \overline{xyz^2} \\
\xrightarrow{b} & \overline{xy^2z^2} & \xrightarrow{a} & \overline{x^2y^2} & \xrightarrow{a} & \overline{y^2z} & \xrightarrow{b} & \overline{z} & \xrightarrow{a} & \overline{xz} & \xrightarrow{b^{-1}} & \overline{xy^2z} \\
\xrightarrow{a} & \overline{x^2y^2z^2} & \xrightarrow{a} & \overline{y^2} & \xrightarrow{b^{-1}} & \overline{y} & \xrightarrow{b^{-1}} & \overline{e} & & & &
\end{array}$$

Its endpoint in G is

$$\begin{aligned}
&a^2ba^{-2}b^2a^2ba^2bab^2a^{-1}ba^2bab^{-1}a^2b^{-2} \\
&\qquad = x^2ywx^{-2}(yw)^2x^2ywx^2ywx(yw)^2x^{-1}ywx^2ywx(yw)^{-1}x^2(yw)^{-2} \\
&\qquad = x^2ywxy^2w^2x^2ywx^2ywxy^2w^2x^2ywx^2ywxy^2w^{-1}x^2yw^{-2}.
\end{aligned}$$

Since the walk is a hamiltonian cycle in G/P, we know that this endpoint is in $P = \langle w \rangle$. So all terms except powers of w must cancel. Thus, we need only calculate the contribution from each appearance of w in this expression. To do this, note that if a term w^i is followed by a net total of j appearances of x, then the term contributes a factor of w^{ir^j} to the product. So the endpoint in G is:

$$w^{r^{13}}w^{2r^{12}}w^{r^{10}}w^{r^8}w^{2r^7}w^{r^5}w^{r^3}w^{-r^2}w^{-2}.$$

Since $r^3 \equiv 1 \pmod{p}$, this simplifies to

$$\begin{aligned}
w^rw^2w^rw^{r^2}w^{2r}w^{r^2}ww^{-r^2}w^{-2} &= w^{r+2+r+r^2+2r+r^2+1-r^2-2} \\
&= w^{r^2+4r+1} = w^{r^2+r+1}w^{3r} = w^0w^{3r} = w^{3r}.
\end{aligned}$$

Since $p \nmid 3r$, this endpoint generates P, so the Factor Group Lemma (1.4) provides a hamiltonian cycle in $\mathrm{Cay}(G;S)$.

Case 2. *Assume $S = \{x, xyw\}$.* For $a = x$ and $b = xyw$, we have the hamiltonian cycle

$$\big((a, b^2)^3\#, a\big)^3$$

in $\mathrm{Cay}(G/P;S)$. Its endpoint in G is

$$\begin{aligned}\left((ab^2)^3\, b^{-1}a\right)^3 &= \Big(\big(x(xyw)^2\big)^3\,(xyw)^{-1}x\Big)^3 = \Big(\big(x(x^2y^2w^{r+1})\big)^3\,(w^{-1}y^{-1}x^{-1})x\Big)^3 \\ &= \Big(\big(y^2w^{r+1}\big)^3\,(w^{-1}y^{-1})\Big)^3 = \Big(w^{3(r+1)}\,(w^{-1}y^{-1})\Big)^3 = \Big(y^{-1}w^{3r+2}\Big)^3 \\ &= w^{3(3r+2)}.\end{aligned}$$

Since we are free to choose r to be either of the two primitive cube roots of 1 in $\mathbb{Z}_p$, and the equation $3r+2=0$ has only one solution in $\mathbb{Z}_p$, we may assume r has been selected to make the exponent nonzero. Then the Factor Group Lemma (1.4) provides a hamiltonian cycle in $\mathrm{Cay}(G;S)$. □

2C. **Sylow 3-subgroup of exponent 9.**

(2.6) **Lemma.** *Assume Q is of exponent 9, so*

$$Q = \langle\, x, y \mid x^9 = y^3 = e,\ [x,y] = x^3 \,\rangle.$$

There are two possibilities for G, depending on whether $C_Q(P)$ contains an element of order 9 or not.

(1) *Assume $C_Q(P)$ does not contain an element of order 9. Then we may assume y centralizes P, but $w^x = w^r$. Furthermore, we may assume:*
 (a) $S = \{x, yw\}$, *or*
 (b) $S = \{x, xyw\}$.
(2) *Assume $C_Q(P)$ contains an element of order 9. Then we may assume x centralizes P, but $w^y = w^r$. Furthermore, we may assume:*
 (a) $S = \{xw, y\}$,
 (b) $S = \{xyw, y\}$,
 (c) $S = \{xy, xw\}$, *or*
 (d) $S = \{xy, x^2yw\}$.

Proof. (1) Since x has order 9, we know that it does not centralize P. But x^3 must centralize P (since x^3 is in G'). Therefore, we may assume $w^x = x^r$ (by replacing x with its inverse if necessary). Also, since $Q/C_Q(P)$ must be cyclic (because $\mathrm{Aut}(P)$ is cyclic), but $C_G(P)$ does not contain an element of order 9, we see that $C_Q(P)$ contains every element of order 3, so y must be in $C_Q(P)$.

Since S must contain an element that does not centralize P, we may assume $x \in S$. By applying Lemma 2.3 with $s = x$ and $c = y$, we see that we may assume S is:

$$\{x, yw\} \text{ or } \{x, y^2w\} \text{ or } \{x, xyw\} \text{ or } \{x, xy^2w\}.$$

The second generating set need not be considered, because $(y^2w)^{-1} = yw^{-1} = yw'$, so it is equivalent to the first. Also, the fourth generating set can be converted into the third, since there is an automorphism of G that fixes y, but takes x to xyw and w to w^{-1}.

(2) We may assume $x \in C_Q(P)$, so $C_Q(P) = \langle x \rangle$.

We know that S must contain an element s that does not centralize P, and there are two possibilities: either

(I) s has order 3, or
(II) s has order 9.

We consider these two possibilities as separate cases.

Case I. *Assume s has order* 3. We may assume $s = y$. Letting $c = x$, we see from Lemma 2.3 that we may assume S is either

$$\{y, xw\} \text{ or } \{y, x^2w\} \text{ or } \{y, yxw\} \text{ or } \{y, yx^2w\}.$$

The second and fourth generating sets need not be considered, because there is an automorphism of G that fixes y and w, but takes x to x^2. Also, the third generating set may be replaced with $\{y, xyw\}$, since there is an automorphism of G that fixes y and w, but takes x to $y^{-1}xy$.

Case II. *Assume s has order* 9. We may assume $s = xy$. Letting $c = x$, we see from Lemma 2.3 that we may assume S is either

$$\{xy, xw\} \text{ or } \{xy, x^2w\} \text{ or } \{xy, xyxw\} \text{ or } \{xy, xyx^2w\}.$$

The second generating set is equivalent to $\{xy, xw\}$, since the automorphism of G that sends x to x^4, y to $x^{-3}y$, and w to w^{-1} maps it to $\{xy, (xw)^{-1}\}$. The third generating set is mapped to $\{xy, x^2yw\}$ by the automorphism that sends x to $x[x, y]$ and y to $[x, y]^{-1}y$. The fourth generating set need not be considered, because xyx^2w is an element of order 3 that does not centralize P, which puts it in the previous case. $\square$

(2.7) **Proposition.** *Assume, as usual, that $|G| = 27p$, where p is prime, and that G has a normal Sylow p-subgroup. If the Sylow 3-subgroup Q is of exponent 9, then $\mathrm{Cay}(G; S)$ has a hamiltonian cycle.*

Proof. We will show that, for an appropriate choice of a and b in $S \cup S^{-1}$, the walk

$$(a^3, b^{-1}, a, b^{-1}, a^4, b^2, a^{-2}, b, a^2, b, a^3, b, a^{-1}, b^{-1}, a^{-1}, b^{-2}) \tag{2.8}$$

provides a hamiltonian cycle in $\mathrm{Cay}(G/P; S)$ whose endpoint in G generates P (so the Factor Group Lemma (1.4) applies).

We begin by verifying two situations in which (2.8) is a hamiltonian cycle:

(HC1) If $|\overline{a}| = 9$, $|\overline{b}| = 3$, and $\overline{a^b} = \overline{a^4}$ in $\overline{G} = G/P$, then we have the hamiltonian cycle

$$\begin{array}{cccccccccccccc}
 & \overline{e} & \xrightarrow{a} & \overline{a} & \xrightarrow{a} & \overline{a^2} & \xrightarrow{a} & \overline{a^3} & \xrightarrow{b^{-1}} & \overline{a^3b^2} & \xrightarrow{a} & \overline{a^7b^2} & \xrightarrow{b^{-1}} & \overline{a^7b} \\
\xrightarrow{a} & \overline{a^5b} & \xrightarrow{a} & \overline{a^3b} & \xrightarrow{a} & \overline{ab} & \xrightarrow{a} & \overline{a^8b} & \xrightarrow{b} & \overline{a^8b^2} & \xrightarrow{b} & \overline{a^8} & \xrightarrow{a^{-1}} & \overline{a^7} \\
\xrightarrow{a^{-1}} & \overline{a^6} & \xrightarrow{b} & \overline{a^6b} & \xrightarrow{a} & \overline{a^4b} & \xrightarrow{a} & \overline{a^2b} & \xrightarrow{b} & \overline{a^2b^2} & \xrightarrow{a} & \overline{a^6b^2} & \xrightarrow{a} & \overline{ab^2} \\
\xrightarrow{a} & \overline{a^5b^2} & \xrightarrow{b} & \overline{a^5} & \xrightarrow{a^{-1}} & \overline{a^4} & \xrightarrow{b^{-1}} & \overline{a^4b^2} & \xrightarrow{a^{-1}} & \overline{b^2} & \xrightarrow{b^{-1}} & \overline{b} & \xrightarrow{b^{-1}} & \overline{e}
\end{array}$$

(HC2) If $|\overline{a}| = 9$, $|\overline{b}| = 9$, $\overline{a^b} = \overline{a^7}$, and $\overline{b^3} = \overline{a^6}$ in $\overline{G} = G/P$, then we have the hamiltonian cycle

$$\begin{array}{cccccccccccccc}
 & \overline{e} & \xrightarrow{a} & \overline{a} & \xrightarrow{a} & \overline{a^2} & \xrightarrow{a} & \overline{a^3} & \xrightarrow{b^{-1}} & \overline{a^6b^2} & \xrightarrow{a} & \overline{a^4b^2} & \xrightarrow{b^{-1}} & \overline{a^4b} \\
\xrightarrow{a} & \overline{a^8b} & \xrightarrow{a} & \overline{a^3b} & \xrightarrow{a} & \overline{a^7b} & \xrightarrow{a} & \overline{a^2b} & \xrightarrow{b} & \overline{a^2b^2} & \xrightarrow{b} & \overline{a^8} & \xrightarrow{a^{-1}} & \overline{a^7} \\
\xrightarrow{a^{-1}} & \overline{a^6} & \xrightarrow{b} & \overline{a^6b} & \xrightarrow{a} & \overline{ab} & \xrightarrow{a} & \overline{a^5b} & \xrightarrow{b} & \overline{a^5b^2} & \xrightarrow{a} & \overline{a^3b^2} & \xrightarrow{a} & \overline{ab^2} \\
\xrightarrow{a} & \overline{a^8b^2} & \xrightarrow{b} & \overline{a^5} & \xrightarrow{a^{-1}} & \overline{a^4} & \xrightarrow{b^{-1}} & \overline{a^7b^2} & \xrightarrow{a^{-1}} & \overline{b^2} & \xrightarrow{b^{-1}} & \overline{b} & \xrightarrow{b^{-1}} & \overline{e}
\end{array}$$

To calculate the endpoint in G, fix $r_1, r_2 \in \mathbb{Z}_p$, with

$$w^a = w^{r_1} \text{ and } w^b = w^{r_2},$$

and write

$$a = \underline{a} w_1 \text{ and } b = \underline{b} w_2, \text{ where } \underline{a}, \underline{b} \in Q \text{ and } w_1, w_2 \in P.$$

Note that if an occurrence of w_i in the product is followed by a net total of j_1 appearances of $\underline{a}$ and a net total of j_2 appearances of $\underline{b}$, then it contributes a factor of $w_i^{r_1^{j_1} r_2^{j_2}}$ to the product. (A similar occurrence of w_i^{-1} contributes a factor of $w_i^{-r_1^{j_1} r_2^{j_2}}$ to the product.) Furthermore, since $r_1^3 \equiv r_2^3 \equiv 1 \pmod p$, there is no harm in reducing j_1 and j_2 modulo 3.

We will apply these considerations only in a few particular situations:

(E1) Assume $w_1 = e$ (so $a \in Q$ and $\underline{a} = a$). Then the endpoint of the path in G is

$$\begin{aligned} &a^3 b^{-1} a b^{-1} a^4 b^2 a^{-2} b a^2 b a^3 b a^{-1} b^{-1} a^{-1} b^{-2} \\ &\quad = a^3 (\underline{b} w_2)^{-1} a (\underline{b} w_2)^{-1} a^4 (\underline{b} w_2)^2 a^{-2} (\underline{b} w_2) a^2 \\ &\qquad \times (\underline{b} w_2) a^3 (\underline{b} w_2) a^{-1} (\underline{b} w_2)^{-1} a^{-1} (\underline{b} w_2)^{-2} \\ &\quad = a^3 (w_2^{-1} \underline{b}^{-1}) a (w_2^{-1} \underline{b}^{-1}) a^4 (\underline{b} w_2 \underline{b} w_2) a^{-2} (\underline{b} w_2) a^2 \\ &\qquad \times (\underline{b} w_2) a^3 (\underline{b} w_2) a^{-1} (w_2^{-1} \underline{b}^{-1}) a^{-1} (w_2^{-1} \underline{b}^{-1} w_2^{-1} \underline{b}^{-1}). \end{aligned}$$

By the above considerations, this simplifies to w_2^m, where

$$\begin{aligned} m &= -1 - r_1^2 r_2 + r_1 r_2 + r_1 + r_2^2 + r_1 r_2 + r_1 - r_1^2 - r_2 - r_2^2 \\ &= -r_1^2 r_2 - r_1^2 + 2 r_1 r_2 + 2 r_1 - r_2 - 1. \end{aligned}$$

Note that:

(a) If $r_1 \neq 1$ and $r_2 = 1$, then m simplifies to $6 r_1$, because $r_1^2 + r_1 + 1 \equiv 0 \pmod p$ in this case.

(b) If $r_1 \neq 1$ and $r_2 \neq 1$, then m simplifies to $3 r_1 (r_2 + 1)$, because $r_1^2 + r_1 + 1 \equiv r_2^2 + r_2 + 1 \equiv 0 \pmod p$ in this case.

(E2) Assume $w_2 = e$ (so $b \in Q$ and $\underline{b} = b$). Then the endpoint of the path in G is

$$\begin{aligned} &a^3 b^{-1} a b^{-1} a^4 b^2 a^{-2} b a^2 b a^3 b a^{-1} b^{-1} a^{-1} b^{-2} \\ &\quad = (\underline{a} w_1)^3 b^{-1} (\underline{a} w_1) b^{-1} (\underline{a} w_1)^4 b^2 (\underline{a} w_1)^{-2} b (\underline{a} w_1)^2 b (\underline{a} w_1)^3 b (\underline{a} w_1)^{-1} b^{-1} (\underline{a} w_1)^{-1} b^{-2} \\ &\quad = (\underline{a} w_1 \underline{a} w_1 \underline{a} w_1) b^{-1} (\underline{a} w_1) b^{-1} (\underline{a} w_1 \underline{a} w_1 \underline{a} w_1 \underline{a} w_1) b^2 (w_1^{-1} \underline{a}^{-1} w_1^{-1} \underline{a}^{-1}) \\ &\qquad \times b (\underline{a} w_1 \underline{a} w_1) b (\underline{a} w_1 \underline{a} w_1 \underline{a} w_1) b (w_1^{-1} \underline{a}^{-1}) b^{-1} (w_1^{-1} \underline{a}^{-1}) b^{-2}. \end{aligned}$$

By the above considerations, this simplifies to w_1^m, where

$$\begin{aligned} m &= r_1^2 + r_1 + 1 + r_1^2 r_2 + r_1 r_2^2 + r_2^2 + r_1^2 r_2^2 + r_1 r_2^2 - r_1 \\ &\quad - r_1^2 + r_1^2 r_2^2 + r_1 r_2^2 + r_2 + r_1^2 r_2 + r_1 r_2 - r_1 - r_1^2 r_2 \\ &= 2 r_1^2 r_2^2 + 3 r_1 r_2^2 + r_2^2 + r_1^2 r_2 + r_1 r_2 + r_2 - r_1 + 1. \end{aligned}$$

Note that:

(a) If $r_1 = 1$ and $r_2 \neq 1$, then m simplifies to $-3(r_2 + 2)$, because $r_2^2 + r_2 + 1 \equiv 0 \pmod p$ in this case.

(b) If $r_1 \neq 1$ and $r_2 \neq 1$, then m simplifies to $-r_1r_2 - 2r_1 + r_2 + 2$, because $r_1^2 + r_1 + 1 \equiv r_2^2 + r_2 + 1 \equiv 0 \pmod{p}$ in this case.

Now we provide a hamiltonian cycle for each of the generating sets listed in Lemma 2.6:

(1a) If $C_Q(P)$ has exponent 3, and $S = \{x, yw\}$, we let $a = x$ and $b = yw$ in (HC1). In this case, we have $w_1 = e$, $r_1 = r$, and $r_2 = 1$, so (E1a) tells us that the endpoint in G is w_2^{6r}.

(1b) If $C_Q(P)$ has exponent 3, and $S = \{x, xyw\}$, we let $a = x$ and $b = (xyw)^{-1}$ in (HC2). In this case, we have $w_1 = e$, $r_1 = r$ and $r_2 = r^{-1} = r^2$, so (E1b) tells us that the endpoint in G is w_2^m, where

$$m = 3r_1(r_2 + 1) = 3r(r^2 + 1) = 3(r^3 + r) \equiv 3(1 + r) = 3(r + 1) \pmod{p}.$$

(2a) If $C_Q(P)$ has exponent 9, and $S = \{xw, y\}$, we let $a = xw$ and $b = y$ in (HC1). In this case, we have $w_2 = e$, $r_1 = 1$ and $r_2 = r$, so (E2a) tells us that the endpoint in G is $w_1^{-3(r+2)}$.

(2b) If $C_Q(P)$ has exponent 9, and $S = \{xyw, y\}$, we let $a = xyw$ and $b = y$ in (HC1). In this case, we have $w_2 = e$ and $r_1 = r_2 = r$, so (E2b) tells us that the endpoint in G is w_2^m, where

$$m = -r_1r_2 - 2r_1 + r_2 + 2 = -r^2 - 2r + r + 2 = -(r^2 + r + 1) + 3 \equiv 3 \pmod{p}.$$

(2c) If $C_Q(P)$ has exponent 9, and $S = \{xy, xw\}$, we let $a = xw$ and $b = (xy)^{-1}$ in (HC2). In this case, we have $w_2 = e$, $r_1 = 1$, and $r_2 = r^{-1} = r^2$, so (E2a) tells us that the endpoint in G is w_1^m, where

$$m = -3(r_2 + 2) = -3(r^2 + 2) \equiv -3\big(-(r + 1) + 2\big) = 3(r - 1) \pmod{p}.$$

(2d) If $C_Q(P)$ has exponent 9, and $S = \{xy, x^2yw\}$, we let $a = xy$ and $b = x^2yw$ in (HC2). In this case, we have $w_1 = e$ and $r_1 = r_2 = r$, so (E1b) tells us that the endpoint in G is w_2^m, where

$$m = 3r_1(r_2 + 1) = 3r(r + 1) = 3(r^2 + r) \equiv 3(-1) = -3 \pmod{p}.$$

In all cases, there is at most one nonzero value of r (modulo p) for which the exponent of w_i is 0. Since we are free to choose r to be either of the two primitive cube roots of 1 in $\mathbb{Z}_p$, we may assume r has been selected to make the exponent nonzero. Then the Factor Group Lemma (1.4) provides a hamiltonian cycle in $\mathrm{Cay}(G; S)$. □

3. Assume the Sylow p-subgroups of G are not normal

(3.1) **Lemma.** *Assume*

- $|G| = 27p$, *where* p *is an odd prime, and*
- *the Sylow* p*-subgroups of* G *are not normal.*

Then $p = 13$, *and* $G = \mathbb{Z}_{13} \ltimes (\mathbb{Z}_3)^3$, *where a generator* w *of* $\mathbb{Z}_{13}$ *acts on* $(\mathbb{Z}_3)^3$ *via multiplication on the right by the matrix*

$$W = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Furthermore, we may assume

$$S \text{ is of the form } \{w^i, w^j v\},$$

where $v = (1,0,0) \in (\mathbb{Z}_3)^3$, *and*

$$(i,j) \in \{(1,0),(2,0),(1,2),(1,3),(1,5),(1,6),(2,5)\}.$$

Proof. Let P be a Sylow p-subgroup of G, and Q be a Sylow 3-subgroup of G. Since no odd prime divides $3-1$ or 3^2-1, and 13 is the only odd prime that divides 3^3-1, Sylow's Theorem [4, Thm. 15.7, p. 230] implies that $p = 13$, and that $N_G(P) = P$, so G must have a normal p-complement [3, Thm. 7.4.3]; i.e., $G = P \ltimes Q$. Since P must act nontrivially on Q (since P is not normal), we know that it must act nontrivially on $Q/\Phi(Q)$ [3, Thm. 5.3.5, p. 180]. However, P cannot act nontrivially on an elementary abelian group of order 3 or 3^2, because $|P| = 13$ is not a divisor of $3-1$ or 3^2-1. Therefore, we must have $|Q/\Phi(Q)| = 3^3$, so Q must be elementary abelian (and the action of P is irreducible).

Let W be the matrix representing the action of w on $(\mathbb{Z}_3)^3$ (with respect to some basis that will be specified later). In the polynomial ring $\mathbb{Z}_3[X]$, we have the factorization:

$$\begin{aligned} \frac{X^{13}-1}{X-1} &= (X^3 - X - 1)\cdot(X^3+X^2-1) \\ &\qquad \cdot (X^3+X^2+X-1)\cdot(X^3-X^2-X-1). \end{aligned} \tag{3.2}$$

Since $w^{13} = e$, the minimal polynomial of W must be one of the factors on the right-hand side. By replacing w with an appropriate power, we may assume it is the first factor. Then, choosing any nonzero $v \in (\mathbb{Z}_3)^3$, the matrix representation of w with respect to the basis $\{v, v^w, v^{w^2}\}$ is W (the Rational Canonical Form).

Now, let ζ be a primitive 13th root of unity in the finite field GF(27). Then any Galois automorphism of GF(27) over GF(3) must raise ζ to a power. Since the subgroup of order 3 in $\mathbb{Z}_{13}^\times$ is generated by the number 3, we conclude that the orbit of ζ under the Galois group is $\{\zeta, \zeta^3, \zeta^9\}$. These must be the 3 roots of one of the irreducible factors on the right-hand side of (3.2). Thus, for any $k \in \mathbb{Z}_{13}^\times$, the matrices W^k, W^{3k}, and W^{9k} all have the same minimal polynomial, so they are conjugate under $\mathrm{GL}_3(3)$. That is:

(3.3) powers of W in the same row of the following table are conjugate under $\mathrm{GL}_3(3)$:

W, W^3, W^9
W^2, W^5, W^6
W^4, W^{12}, W^{10}
W^7, W^8, W^{11}

There is an element a of S that generates $G/Q \cong P$. Then a has order p, so, replacing it by a conjugate, we may assume $a \in P = \langle w \rangle$, so $a = w^i$ for some $i \in \mathbb{Z}_{13}^\times$. From (3.3), we see that we may assume $i \in \{1,2\}$ (perhaps after replacing a by its inverse).

Now let b be the second element of S, so we may assume $b = w^j v$ for some j. We may assume $0 \le j \le 6$ (by replacing b with its inverse, if necessary). We may also assume $j \ne i$, for otherwise $S \subset aQ$, so Theorem 1.8 applies.

If $j = 0$, then (i,j) is either $(1,0)$ or $(2,0)$, both of which appear in the list; henceforth, let us assume $j \neq 0$.

Case 1. *Assume* $i = 1$. Since $j \neq i$, we must have $j \in \{2,3,4,5,6\}$.

Note that, since W^3 is conjugate to W under $\mathrm{GL}_3(3)$ (since they are in the same row of (3.3)), we know that the pair (w, w^4) is isomorphic to the pair $\big(w^3, (w^3)^4\big) =$

(w^3, w^{-1}). By replacing b with its inverse, and then interchanging a and b, this is transformed to (w, w^3). So we may assume $j \neq 4$.

Case 2. *Assume $i = 2$.* We may assume W^j is in the second or fourth row of the table (for otherwise we could interchange a with b to enter the previous case. So $j \in \{2, 5, 6\}$. Since $j \neq i$, this implies $j \in \{5, 6\}$. However, since W^5 is conjugate to W^2 (since they are in the same row of (3.3)), and we have $(w^2)^3 = w^6$ and $(w^5)^3 = w^2$, we see that the pair (w^2, w^6) is isomorphic to (w^2, w^5). So we may assume $j \neq 6$. □

(3.4) **Proposition.** *If $|G| = 27p$, where p is prime, and the Sylow p-subgroups of G are not normal, then* $\mathrm{Cay}(G; S)$ *has a hamiltonian cycle.*

Proof. From Lemma 3.1 (and Remark 1.9), we may assume $G = \mathbb{Z}_{13} \ltimes (\mathbb{Z}_3)^3$. For each of the generating sets listed in Lemma 3.1, we provide an explicit hamiltonian cycle in the quotient multigraph $P\backslash\mathrm{Cay}(G; S)$ that uses at least one double edge. So Lemma 1.7 applies.

To save space, we use $i_1 i_2 i_3$ to denote the vertex $P(i_1, i_2, i_3)$.

$(i, j) = (1, 0)$ $a = w$, $a^{-1} = w^{12}$, $b = (1, 0, 0)$, $b^{-1} = (-1, 0, 0)$
Double edge: $222 \to 022$ with a^{-1} and b

$$\begin{array}{l} 000 \xrightarrow{b^{-1}} 200 \xrightarrow{a} 020 \xrightarrow{a} 002 \xrightarrow{a} 220 \xrightarrow{b^{-1}} 120 \xrightarrow{a} 012 \\ \xrightarrow{a} 221 \xrightarrow{a} 102 \xrightarrow{b} 202 \xrightarrow{a} 210 \xrightarrow{a} 021 \xrightarrow{a} 112 \xrightarrow{a} 201 \\ \xrightarrow{b^{-1}} 101 \xrightarrow{a^{-1}} 211 \xrightarrow{a^{-1}} 212 \xrightarrow{a^{-1}} 222 \xrightarrow{b} 022 \xrightarrow{b} 122 \xrightarrow{a^{-1}} 121 \\ \xrightarrow{a^{-1}} 111 \xrightarrow{b^{-1}} 011 \xrightarrow{a^{-1}} 110 \xrightarrow{a^{-1}} 001 \xrightarrow{a^{-1}} 010 \xrightarrow{a^{-1}} 100 \xrightarrow{b^{-1}} 000 \end{array}$$

$(i, j) = (2, 0)$ $a = w^2$, $a^{-1} = w^{11}$, $b = (1, 0, 0)$, $b^{-1} = (-1, 0, 0)$
Double edge: $020 \to 220$ with a and b^{-1}

$$\begin{array}{l} 000 \xrightarrow{b^{-1}} 200 \xrightarrow{a} 002 \xrightarrow{a} 022 \xrightarrow{a} 212 \xrightarrow{b^{-1}} 112 \xrightarrow{a^{-1}} 210 \\ \xrightarrow{a^{-1}} 122 \xrightarrow{a^{-1}} 111 \xrightarrow{a^{-1}} 110 \xrightarrow{b^{-1}} 010 \xrightarrow{a^{-1}} 201 \xrightarrow{b^{-1}} 101 \xrightarrow{a} 012 \\ \xrightarrow{a} 102 \xrightarrow{a} 020 \xrightarrow{b^{-1}} 220 \xrightarrow{a} 222 \xrightarrow{a} 211 \xrightarrow{a} 120 \xrightarrow{a} 221 \\ \xrightarrow{b} 021 \xrightarrow{a^{-1}} 202 \xrightarrow{a^{-1}} 121 \xrightarrow{a^{-1}} 011 \xrightarrow{a^{-1}} 001 \xrightarrow{a^{-1}} 100 \xrightarrow{b^{-1}} 000 \end{array}$$

$(i, j) = (1, 2)$ $a = w$, $a^{-1} = w^{12}$, $b = w^2(1, 0, 0)$, $b^{-1} = w^{11}(-1, -1, 1)$
Double edge: $220 \to 022$ with a and b

$$\begin{array}{l} 000 \xrightarrow{b^{-1}} 221 \xrightarrow{a^{-1}} 012 \xrightarrow{a^{-1}} 120 \xrightarrow{b^{-1}} 102 \xrightarrow{b^{-1}} 200 \xrightarrow{a} 020 \\ \xrightarrow{a} 002 \xrightarrow{a} 220 \xrightarrow{b} 022 \xrightarrow{a} 222 \xrightarrow{b} 011 \xrightarrow{a} 111 \xrightarrow{a} 121 \\ \xrightarrow{a} 122 \xrightarrow{a} 202 \xrightarrow{a} 210 \xrightarrow{a} 021 \xrightarrow{a} 112 \xrightarrow{b^{-1}} 101 \xrightarrow{a^{-1}} 211 \\ \xrightarrow{a^{-1}} 212 \xrightarrow{b} 201 \xrightarrow{b} 110 \xrightarrow{a^{-1}} 001 \xrightarrow{a^{-1}} 010 \xrightarrow{a^{-1}} 100 \xrightarrow{b^{-1}} 000 \end{array}$$

$(i,j) = (1,3) \quad a = w, \quad a^{-1} = w^{12}, \quad b = w^3(1,0,0), \quad b^{-1} = w^{10}(0,1,-1)$
Double edge: $200 \to 020$ with a and b

$$\begin{array}{l} 000 \xrightarrow{b^{-1}} 012 \xrightarrow{a^{-1}} 120 \xrightarrow{b^{-1}} 221 \xrightarrow{a} 102 \xrightarrow{a} 200 \xrightarrow{b} 020 \\ \xrightarrow{a} 002 \xrightarrow{a} 220 \xrightarrow{a} 022 \xrightarrow{a} 222 \xrightarrow{a} 212 \xrightarrow{a} 211 \xrightarrow{a} 101 \\ \xrightarrow{b^{-1}} 201 \xrightarrow{a^{-1}} 112 \xrightarrow{a^{-1}} 021 \xrightarrow{a^{-1}} 210 \xrightarrow{a^{-1}} 202 \xrightarrow{a^{-1}} 122 \xrightarrow{b} 121 \\ \xrightarrow{a^{-1}} 111 \xrightarrow{a^{-1}} 011 \xrightarrow{a^{-1}} 110 \xrightarrow{a^{-1}} 001 \xrightarrow{a^{-1}} 010 \xrightarrow{a^{-1}} 100 \xrightarrow{b^{-1}} 000 \end{array}$$

$(i,j) = (1,5) \quad a = w, \quad a^{-1} = w^{12}, \quad b = w^5(1,0,0), \quad b^{-1} = w^{8}(1,0,1)$
Double edge: $220 \to 022$ with a and b^{-1}

$$\begin{array}{l} 000 \xrightarrow{b^{-1}} 101 \xrightarrow{a} 120 \xrightarrow{a} 012 \xrightarrow{a} 221 \xrightarrow{b^{-1}} 010 \xrightarrow{a} 001 \\ \xrightarrow{a} 110 \xrightarrow{a} 011 \xrightarrow{a} 111 \xrightarrow{b} 121 \xrightarrow{a} 122 \xrightarrow{b^{-1}} 102 \xrightarrow{a} 200 \\ \xrightarrow{a} 020 \xrightarrow{a} 002 \xrightarrow{a} 220 \xrightarrow{b^{-1}} 022 \xrightarrow{a} 222 \xrightarrow{a} 212 \xrightarrow{a} 211 \\ \xrightarrow{b} 202 \xrightarrow{a} 210 \xrightarrow{a} 021 \xrightarrow{a} 112 \xrightarrow{a} 201 \xrightarrow{a} 100 \xrightarrow{b^{-1}} 000 \end{array}$$

$(i,j) = (1,6) \quad a = w, \quad a^{-1} = w^{12}, \quad b = w^6(1,0,0), \quad b^{-1} = w^{7}(-1,1,1)$
Double edge: $021 \to 210$ with a^{-1} and b

$$\begin{array}{l} 000 \xrightarrow{b^{-1}} 211 \xrightarrow{b^{-1}} 201 \xrightarrow{a^{-1}} 112 \xrightarrow{a^{-1}} 021 \xrightarrow{b} 210 \xrightarrow{b} 101 \\ \xrightarrow{b} 120 \xrightarrow{a} 012 \xrightarrow{a} 221 \xrightarrow{a} 102 \xrightarrow{a} 200 \xrightarrow{a} 020 \xrightarrow{a} 002 \\ \xrightarrow{a} 220 \xrightarrow{a} 022 \xrightarrow{a} 222 \xrightarrow{a} 212 \xrightarrow{b} 202 \xrightarrow{a^{-1}} 122 \xrightarrow{a^{-1}} 121 \\ \xrightarrow{a^{-1}} 111 \xrightarrow{a^{-1}} 011 \xrightarrow{a^{-1}} 110 \xrightarrow{a^{-1}} 001 \xrightarrow{a^{-1}} 010 \xrightarrow{a^{-1}} 100 \xrightarrow{b^{-1}} 000 \end{array}$$

$(i,j) = (2,5) \quad a = w^2, \quad a^{-1} = w^{11}, \quad b = w^5(1,0,0), \quad b^{-1} = w^{8}(1,0,1)$
Double edge: $112 \to 210$ with a^{-1} and b

$$\begin{array}{l} 000 \xrightarrow{b^{-1}} 101 \xrightarrow{a} 012 \xrightarrow{b} 102 \xrightarrow{a} 020 \xrightarrow{a} 220 \xrightarrow{a} 222 \\ \xrightarrow{b} 112 \xrightarrow{b} 210 \xrightarrow{a^{-1}} 122 \xrightarrow{a^{-1}} 111 \xrightarrow{a^{-1}} 110 \xrightarrow{a^{-1}} 010 \xrightarrow{a^{-1}} 201 \\ \xrightarrow{a^{-1}} 021 \xrightarrow{a^{-1}} 202 \xrightarrow{b^{-1}} 211 \xrightarrow{a} 120 \xrightarrow{a} 221 \xrightarrow{a} 200 \xrightarrow{a} 002 \\ \xrightarrow{a} 022 \xrightarrow{a} 212 \xrightarrow{b^{-1}} 121 \xrightarrow{a^{-1}} 011 \xrightarrow{a^{-1}} 001 \xrightarrow{a^{-1}} 100 \xrightarrow{b^{-1}} 000 \end{array}$$ □

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